MTH 508/508: Midterm solutions

1. For $n \geq 1$, consider the *unitary group* defined by

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) : A^* = A^{-1}\}$$

and the *special unitary group* defined by

$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) : \det(A) = 1.\}$$

(a) Show that $\mathrm{U}(n)$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$ of dimension n^2.

(b) Show that $\mathrm{SU}(n)$ is a compact Lie subgroup of $\mathrm{U}(n)$ of dimension $n^2 - 1$.

Solution. (a) Since $\mathbb{C}$ is a 2-dimensional smooth manifold and $\mathrm{GL}(n, \mathbb{R})$ is an n^2-dimensional Lie group, it follows that $\mathrm{GL}(n, \mathbb{C})$ is a $2n^2$-dimensional smooth Lie group (Verify this!). Consider the map $f : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ defined by $f(A) = AA^*$. Clearly, f is smooth as all of its component functions are polynomials. Moreover, f has constant rank n^2 in $\mathrm{GL}(n, \mathbb{C})$ (Verify this!). By the Regular Value Theorem, we have $F^{-1}(\{I_n\}) = \{A \in \mathrm{GL}(n, \mathbb{C}) : AA^* = I_n\} = \mathrm{U}(n)$ is an n^2-dimensional regular closed submanifold of $\mathrm{GL}(n, \mathbb{C})$. By Theorem 1.3.1 (iv) and fact that $\mathrm{U}(n) < \mathrm{GL}(n, \mathbb{C})$, it follows that $\mathrm{U}(n)$ is a Lie subgroup. Furthermore, we note that $U(n)$ is a subspace of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$, which is endowed with a metric induced by the matrix norm $\|A\| = \sqrt{\mathrm{tr}(A^*A)}$. Thus, for any $A \in U(n)$, we have

$$\|A\| = \sqrt{\mathrm{tr}(A^*A)} = \sqrt{\mathrm{tr}(I_n)} = \sqrt{n},$$

which shows $U(n)$ is bounded. Since $U(n)$ is a closed and bounded subspace of an Euclidean space $M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$, it is compact.

(b) First, we note that smooth map $\det : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^*$ has constant rank 1 (Verify this!). Consequently, $\det|_{U(n)} : U(n) \to \mathbb{C}^*$ has constant rank 1. By the Regular Value Theorem, Theorem 1.3.1 (iv) and the fact that $\mathrm{SU}(n) < \mathrm{U}(n)$, it follows that $\mathrm{SU}(n) = \det^{-1}(\{1\})$ is a closed submanifold and a Lie subgroup of $\mathrm{U}(n)$ of dimension n^2-1. Since $U(n)$ is compact, it follows that $\mathrm{SU}(n)$ is compact.

2. The complex projective n-space, denoted by $\mathbb{C}P^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$.

(a) Show that π is a smooth submersion.

(b) Show that $\mathbb{C}P^n$ is a compact $2n$-dimensional smooth manifold. [Hint: Recall the smooth structure in $\mathbb{R}P^n$.]

(c) Show that that the map $G : \mathbb{C}^n \to \mathbb{C}P^n$ defined by

$$G(z_1, \ldots, z_n) = [z_1, \ldots, z_n, 1]$$

is a diffeomorphism onto a dense subset of $\mathbb{C}P^n$.

Solution. (b) Note that $\mathbb{C}P^n$ is Hausdorff and second-countable (Verify this!). As in case of $\mathbb{R}P^n$, the complex projective n-space $\mathbb{C}P^n$ is a differentiable n^2-manifold with the structure determined by the coordinate neighborhoods $\{(U_i, \varphi_i) : 1 \leq i \leq n+1\}$, where:

$$U_i = \{\pi(\bar{U}_i) : \bar{U}_i = \{x \in \mathbb{C}^{n+1} (\cong \mathbb{R}^{2n+2}) : x_i \neq 0\}\}$$

and $\varphi_i : U_i \to \mathbb{C}^n (\cong \mathbb{R}^{2n})$ is defined by

$$\varphi_i(z_1, \dots z_{n+1}) = \left(\frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_{n+1}}{z_i} \right).$$

Though, for convenience, these neighborhoods have been described in complex coordinates, it is straightforward to express them in real coordinates. (Verify this!)

It remains to prove the compactness of $\mathbb{C}P^n$. First, we note that there is a natural properly discontinuous action $S^1 \times S^{2n+1} (\subset \mathbb{C}^n) \to S^{2n+1}$ given by $(e^{i\theta}, (z_1, \dots, z_n)) \mapsto (e^{i\theta} z_1, \dots e^{i\theta} z_n)$. We see that $\mathbb{C}P^n$ is orbit space of this action, that is, $\mathbb{C}P^n \approx S^{2n+1}/S^1$ (Verify this!). Since the induced quotient map $S^{2n+1} \to \mathbb{C}P^n$ is continuous and S^{2n+1} is compact, it follows that $\mathbb{C}P^n$ is compact.

(a) For $p = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$, let $\pi(p) \in U_i$ for some i so that

$$(\varphi_i \circ \pi)(p) = \left(\frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_{n+1}}{z_i} \right).$$

Then the Jacobian $D(\varphi_i \circ \pi)(p)$ (of complex partial derivatives) at p is a matrix $n \times (n+1)$ complex matrix in that has rank n (Verify this!). Therefore, π is a smooth submersion.

(c) First, we establish the smoothness of G. For each $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we consider the coordinate neighborhood $(\mathbb{C}^n, id_{\mathbb{C}^n})$ and coordinate neighborhood of $G(z) = [z_1, \dots, z_n, 1]$ given by (U_{n+1}, φ_{n+1}), where $U_{n+1} = \{[w_1, \dots, w_{n+1}] \in \mathbb{C}P^n : w_{n+1} \neq 0\})$ and $\varphi_{n+1}([w_1, \dots, w_{n+1}]) = (\frac{w_1}{w_{n+1}}, \dots, \frac{w_n}{w_{n+1}}) \in \mathbb{C}^n$. Then, we see that $\varphi_{n+1} \circ G \circ id_{\mathbb{C}^n}(w_1, \dots, w_n) = (w_1, \dots, w_n)$, which is clearly smooth. Hence, it follows that G is smooth. Moreover, we have that $G^{-1}([w_1, \dots, w_{n+1}]) = (\frac{w_1}{w_{n+1}}, \dots, \frac{w_n}{w_{n+1}})$, which is smooth. Thus, G is a diffeomorphism.

To show that $G(\mathbb{C}^n)$ is dense, it suffices to show that $\mathbb{C}P^n \setminus G(\mathbb{C}^n) \subset \overline{G(\mathbb{C}^n)}$. Consider a typical point $[z_1, \dots, z_n, 0] \in \mathbb{C}P^n \setminus G(\mathbb{C}^n)$ (Why is a typical point of this form?). Since $(z_1, \dots, z_n, 1/n) \to (z_1, \dots, z_n, 0)$ in $\mathbb{C}^{n+1}$ and π is continuous, it follows that: $[z_1, \dots, z_n, 1/k] \to [z_1, \dots, z_n, 0]$. But we have

$$[z_1, \dots, z_n, 1/k] = [kz_1, \dots, kz_n, 1] = G(kz_1, \dots, kz_n),$$

which implies that $G(kz_1, \dots, kz_n) \to [z_1, \dots, z_n, 0]$. Therefore, it follows that $[z_1, \dots, z_n, 0] \in \overline{G(\mathbb{C}^n)}$.

3. For any $X \in \mathrm{GL}(n, \mathbb{R})$, show that $T_X(\mathrm{GL}(n, \mathbb{R})) \cong M_n(\mathbb{R})$. [Hint: Use the continuity of $\det : M_n(\mathbb{R}) \to \mathbb{R}$ to find a path $\gamma : (-\epsilon, \epsilon) \to \mathrm{GL}(n, \mathbb{R})$ such that $\gamma(0) = I_n$ and $\gamma'(t) = B \in M_n(\mathbb{R})$.]

   Solution. It suffices to determine the tangent space at $I_n \in \mathrm{GL}(n, \mathbb{R})$ (Why). Since $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous, there exists $\epsilon > 0$ such that for all $A \in M_n(\mathbb{R})$ with $\|A\| < \epsilon$, we have $\det(I_n + H) \neq 0$. Hence, given a matrix $A \in M_n(\mathbb{R})$, there exits $\epsilon_A > 0$ such that for $t \in (-\epsilon_A, \epsilon_A)$, we have $\det(I_n + tA) \neq 0$. Now, for $A \in M_n(\mathbb{R})$, consider the smooth path $\gamma : (-\epsilon_A, \epsilon_A) \to \mathrm{GL}(n, \mathbb{R})$ given by $\gamma(t) = I_n + tA$. Then $\gamma(0) = I_n$ and $\gamma'(0) = A \in M_n(\mathbb{R})$. Since $\mathrm{GL}(n, \mathbb{R})$ is an n^2-dimensional smooth submanifold of $M_n(\mathbb{R}) \approx \mathbb{R}^{n^2}$, it follows from 1.1.2(vii) of the Lesson Plan that $T_{I_n}(\mathrm{GL}(n, \mathbb{R})) \cong M_n(\mathbb{R})$ (Verify this!).

4. Show that there exists a smooth vector field on S^2 that vanishes at exactly one point.

Solution. Let N, S denote the north and south poles of S^2. Let $\varphi_1 : S^2 - \{N\} \to \mathbb{R}^2$ (resp. $\varphi_2 : S^2 - \{N\} \to \mathbb{R}^2$) be the stereographic projections from the north (resp. south) poles of S^2. First, we note that $(S^2 - \{N\}, \varphi_1)$ and $(S^2 - \{S\}, \varphi_2)$ determine a smooth structure on S^2 (Verify this!). For $i = 1, 2$, let (u_i, v_i) represent the stereographic coordinates with respect φ_i. We consider the basis element $\frac{\partial}{\partial u_1}$ of $\mathbb{R}^2$. Using the change of basis formula in 2.1 (xii) (a) of the Lesson Plan, we have

$$\frac{\partial}{\partial u_1} = (v_2^2 - u_2^2)\frac{\partial}{\partial u_2} - (2u_2v_2)\frac{\partial}{\partial v_2}.$$

(Verify this!). We now define a vector field $X : S^2 \to \mathbb{R}^2$ defined by $X(p) = X_p$, where:

$$X_p = \begin{cases} (\varphi_1)_*^{-1}\left(\frac{\partial}{\partial u_1}\right), & \text{if } p \in S^2 - \{N\}, \text{ and} \\ (\varphi_2)_*^{-1}\left((v_2^2 - u_2^2)\frac{\partial}{\partial u_2} - (2u_2v_2)\frac{\partial}{\partial v_2}\right), & \text{if } p \in S^2 - \{S\}. \end{cases}$$

Note that the vector field X is smooth, $X_N = 0$, and $X_p \neq 0$ for $p \in S^2 \setminus \{N\}$, as desired. (Verify this!)